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Effective elastic moduli of composites reinforced by particle or fiber with an inhomogeneous interphase

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Abstract

The solution of the strain energy change of an infinite matrix due to the presence of one spherical particle or cylindrical fiber surrounded by an inhomogeneous interphase is the basis of solving effective elastic moduli of corresponding composites based on various micromechanics models. In order to find out the strain energy change, the composite sphere or cylinder, i.e., the spherical particle or cylindrical fiber together with its interphase, is replaced by an effective homogeneous particle or fiber. Independent governing differential equations for each modulus of the effective particle or fiber are derived by extending the replacement method [J. Mech. Phys. Solids 12 (1964) 199]. As far as the strain energy changes of the infinite matrix subjected to various far-field stress systems are concerned, the present model is simple. Meanwhile, FEM analysis is carried out for a verification, which shows that the model can lead to rather accurate results for most practical interphases. Besides, to check the validity of the model further when the interactions among composite cylinders exist, the two problems of an infinite matrix containing two composite cylinders and the effective moduli of composites with the equilateral triangular distribution of composite cylinders are analyzed using FEM. The FEM results show that the model is still rather accurate, especially for the case of interphase properties varying between those of fiber and matrix. Therefore, composite spheres or cylinders are assumed as the effective homogeneous particles or fibers and simple expressions of the effective moduli of composites containing the composite spheres or cylinders are obtained. Furthermore, the present model is compared with some existing models that are based on very complicated derivations.

Keywords: Effective moduli; Inhomogeneous interphase; Composite sphere/cylinder; Effective particle/fiber; Strain energy changes; Replacement method; Independent governing differential equations

1. Introduction

Effective properties of two-phase composites have been extensively studied, and various micromechanics models have been proposed (see the reviews by Willis, 1981; Hashin, 1983; Christensen, 1990; Weng, 1984, 1990; Zimmerman, 1991, 1996; Nemat-Nasser and Hori, 1993; Kachanov, 1992, 1994; Ju and Chen, 1994;

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Huang et al., 1994a,b). It is known that the elastic solution of an infinite matrix containing one inclusion is the basis of these micromechanics models. Therefore, in order to use the micromechanics theories to obtain the effective properties of composites involving a coating on particle/fiber or an inhomogeneous interphase between particle/fiber and matrix, ones need to find the solution of the single inclusion problem involving a coating or an inhomogeneous interphase (Hill, 1964; Hashin, 1972; Qiu and Weng, 1991; Jasiuk and Kouider, 1993; Lutz and Zimmerman, 1996). Compared with the problem involving an inhomogeneous interphase, the case of homogeneous coating is much easier, and some explicit solutions associated with spherical particle or cylindrical fiber exist (Hill, 1964; Hashin and Rosen, 1964; Qiu and Weng, 1991). But these solutions are rather lengthy when the far-field stresses are pure-shear. For the case of inhomogeneous interphase, numerical procedures are usually required to solve the governing differential equations for the elasticity problem of an infinite matrix containing one particle/fiber with the interphase (see Jasiuk and Kouider, 1993). For a very special case, i.e., the radially symmetric problem of an infinite body containing a composite sphere, Lutz and Zimmerman (1996) gave an infinite series form of solution, and then derived the solution for the effective bulk modulus based on the Mori–Tanaka method. Ding and Weng (1999) derived the bulk moduli of some particle- and fiber-reinforced composites with inhomogeneous matrix, which is equivalent to the present case without matrix. But the corresponding problems involving a far-field shear boundary condition have not been analytically or numerically solved.

For composites reinforced by a coated fiber, Hill (1964) pointed out that the fiber together with its surrounding coating could be replaced by a homogeneous fiber when solving for the effective moduli, except for the effective in-plane shear modulus. And the elastic properties of the effective homogeneous fiber can be determined by a solution of effective moduli of two-phase composites. The solution is actually equivalent to the composite cylinder assemblage model (Hashin and Rosen, 1964), the Mori–Tanaka solution (Weng, 1984), or the generalized non-interacting solution (Shen and Yi, 2001).

It is noted that the replacement method is very simple, whereas existing models for the case of inhomogeneous interphase are very complicated. Thus, the authors attempt to extend the simple replacement method so that the complicated cases involving shear boundary stresses and inhomogeneous interphase can be covered. As a result, a set of independent governing differential equations is derived that comprises the key part of the present model. In general, the model is an approximation for the complicated cases. Therefore, a detailed comparison with FEM analysis is carried out to check its validity. Then, some useful results are obtained, including a very simple expression for the effective moduli of composites with inhomogeneous interphases for most practical cases.

2. Two-phase composites

Some notations and useful results for two-phase composites are first introduced that will be used throughout the study. Besides, the isotropic and transversely isotropic situations are considered for the cases associated with spherical particles and unidirectional and continuous fibers, respectively. Therefore, the bulk and shear moduli denoted as K and G can be chosen to characterize the material properties for the case of particles; whereas the longitudinal Young's modulus, major Poisson's ratio, axial shear modulus, plane-strain bulk modulus, and transverse shear modulus denoted as E_{11}, ν_{12}, G_{12}, K_{23}, G_{23} by taking direction 1 as the symmetry axis are chosen for the case of fibers. The superscripts m, f or i will be used to designate the properties corresponding to matrix, fiber/particle or interphase in terms of the context. As the effective longitudinal Young's modulus and major Poisson's ratio of the concerned composites can be satisfactorily predicted by the law of mixtures, the focus is put on the five other moduli including K and G for particle-reinforced composites and G_{12}, K_{23}, G_{23} for fiber-reinforced ones. The five effective elastic moduli will be uniformly treated in the study.

2.1. The strain energy changes of an infinite matrix

The strain energy changes of an infinite matrix due to the presence of one single spherical particle or continuous cylindrical fiber are first analyzed as a basic problem. Corresponding to the five moduli, i.e., K, G, G_{12}, K_{23}, G_{23}, the five far-field stress systems are considered as follows:

$$\sigma_{11}^{\infty} = \sigma_{22}^{\infty} = \sigma_{33}^{\infty} = \sigma \tag{1a}$$

$$\sigma_{23}^{\infty} = \sigma_{32}^{\infty} = \sigma \tag{1b}$$

$$\sigma_{12}^{\infty} = \sigma_{21}^{\infty} = \sigma \tag{1c}$$

$$\sigma_{22}^{\infty} = \sigma_{33}^{\infty} = \sigma, \quad \sigma_{11}^{\infty} \text{ that leads to plane strain condition} \tag{1d}$$

$$\sigma_{23}^{\infty} = \sigma_{32}^{\infty} = \sigma \tag{1e}$$

where the other stress components vanish. In terms of Eshelby's method (1957), the five strain energy changes Δf_C of the infinite matrix subjected to the five far-field stress systems can be uniformly given as

$$\Delta f_C = -\frac{1}{2} V^{\mathrm{f}} \frac{(\sigma)^2}{C^{\mathrm{m}}} \frac{C^{\mathrm{f}} - C^{\mathrm{m}}}{C^{\mathrm{m}} + \alpha_C^{\mathrm{m}}(C^{\mathrm{f}} - C^{\mathrm{m}})} \tag{2}$$

where C and V^{f} denote one of the five moduli, i.e., K, G, G_{12}, K_{23}, G_{23}, and the volume of the particle or that of the fiber per length, and the parameter α_C^{m} corresponding to the five cases is given by

$$\alpha_K^{\mathrm{m}} = \frac{1 + \nu^{\mathrm{m}}}{3(1 - \nu^{\mathrm{m}})} \tag{3a}$$

$$\alpha_G^{\mathrm{m}} = \frac{8 - 10\nu^{\mathrm{m}}}{15(1 - \nu^{\mathrm{m}})} \tag{3b}$$

$$\alpha_{G_{12}}^{\mathrm{m}} = 2 \tag{3c}$$

$$\alpha_{K_{23}}^{\mathrm{m}} = \frac{1}{2(1 - \nu_{23}^{\mathrm{m}})} \tag{3d}$$

$$\alpha_{G_{23}}^{\mathrm{m}} = \frac{3 - 4\nu_{23}^{\mathrm{m}}}{4(1 - \nu_{23}^{\mathrm{m}})} \tag{3e}$$

2.2. The generalized non-interacting solution

Shen and Yi (2001) have proposed a different energy balance equation from the traditional one and derived the generalized non-interacting solution for effective moduli of composites containing ellipsoidal inhomogeneities, which coincides with the Mori–Tanaka solution (Weng, 1984, 1990) for the case of circular (2-D) or spherical (3-D) inhomogeneities. By considering the notations in (3a–e), the generalized non-interacting solution for the effective moduli of two-phase composites corresponding to the present concerned five cases can be uniformly written as

$$\frac{\overline{C} - C^{\mathrm{m}}}{C^{\mathrm{m}} + \alpha_C^{\mathrm{m}}(\overline{C} - C^{\mathrm{m}})} = \phi \frac{C^{\mathrm{f}} - C^{\mathrm{m}}}{C^{\mathrm{m}} + \alpha_C^{\mathrm{m}}(C^{\mathrm{f}} - C^{\mathrm{m}})} \tag{4}$$

or

$$\frac{\overline{C}}{C^{\mathrm{m}}} = 1 + \frac{\phi}{C^{\mathrm{m}}/[C^{\mathrm{f}} - C^{\mathrm{m}}] + (1-\phi)\alpha_C^{\mathrm{m}}} \tag{5}$$

where $\overline{C}$ and ϕ denote the effective moduli and volume fraction of spherical particles or unidirectional continuous fibers. Note that the left side of (4) is associated with the strain energy change of an infinite matrix due to the presence of a spherical/cylindrical RVE of composites, and the right side is associated with that of particles/fibers with volume fraction being ϕ when neglecting the interactions among these particles/fibers.

3. Composites involving an inhomogeneous interphase

3.1. An extension of the replacement method

The replacement method (see Hill, 1964; Hashin, 1972; Qiu and Weng, 1991) has been used to find the effective moduli such as $\overline{K}$ or $\overline{E}_{11}$, $\bar{v}_{12}$, $\overline{G}_{12}$, $\overline{K}_{23}$ of the particle or fiber-reinforced composites with an interphase that can be homogeneous or multi-layered. For the two complicated cases of $\overline{G}$ and $\overline{G}_{23}$, one usually finds them by solving the corresponding elasticity problems (Hashin, 1972; Qiu and Weng, 1991). For the continuously varying interphase properties, one also needs to find each effective modulus by solving their corresponding elasticity problems. For example, Jasiuk and Kouider (1993) obtained $\overline{G}_{12}$, $\overline{K}_{23}$, $\overline{G}_{23}$ by numerically solving the coupled differential equations governing the displacements of the elasticity problem. Lutz and Zimmerman (1996) gave a series form of solution for $\overline{K}$ by solving the radially symmetric problem of an infinite body containing a composite sphere. It can be seen that these approaches are very complicated. Besides, the analytical or numerical results for the more complicated case of an infinite matrix containing one single composite sphere with a varying interphase has not been reported when it is subjected to shear boundary stresses.

In this study, the replacement method is extended to the case of varying interphase properties, as well as the two complicated cases of $\overline{G}$ and $\overline{G}_{23}$, based on the generalized non-interacting solution (5).

A particle/fiber together with its surrounding interphase is called a composite sphere/cylinder. It is assumed that there exists a homogeneous particle or fiber with the same size that induces the same strain energy change of the infinite matrix as that induced by the composite sphere/cylinder. Let C^{eff} denote the corresponding elastic properties of the effective homogeneous particle/fiber that will be exactly or approximately obtained by extending the replacement method.

Let r_0 and r_1 denote the radii of the particle/fiber and the composite sphere/cylinder, respectively, and $C^{\mathrm{eff}}(r)$ be the effective properties of the composite sphere/cylinder with radius $r \in [r_0, r_1]$. When considering a small incremental layer of the interphase from r to $r + \mathrm{d}r$, and assuming the small layer as a homogeneous layer with properties being $C^{\mathrm{i}}(r + \theta\,\mathrm{d}r)$ (where θ is an arbitrary fraction, $\theta \in [0, 1]$) and the surrounded composite sphere in the small layer as an effective homogeneous particle/fiber with $C^{\mathrm{eff}}(r)$, the effective properties of the composite sphere/cylinder with radius $r + \mathrm{d}r$ is obtained using the two-phase formulas (5) as follows:

$$C^{\mathrm{eff}}(r+\mathrm{d}r) = C^{\mathrm{i}}(r+\theta\,\mathrm{d}r) + \frac{\phi C^{\mathrm{i}}(r+\theta\,\mathrm{d}r)}{C^{\mathrm{i}}(r+\theta\,\mathrm{d}r)/[C^{\mathrm{eff}}(r) - C^{\mathrm{i}}(r+\theta\,\mathrm{d}r)] + (1-\phi)\alpha_C^{\mathrm{i}}(r+\theta\,\mathrm{d}r)} \tag{6}$$

where ϕ is the volume fraction of the composite sphere/cylinder with radius $r \in [r_0, r_1]$ when considering the small layer as matrix, i.e., $\phi = r^3/(r+\mathrm{d}r)^3$ for the case of a composite sphere or $\phi = r^2/(r+\mathrm{d}r)^2$ for the case of a composite cylinder. Then, considering the limiting procedure of dr approaching zero, a governing differential equation for $C^{\mathrm{eff}}(r)$ is derived as

$$\frac{\mathrm{d}C^{\mathrm{eff}}(r)}{\mathrm{d}r} = -\frac{m}{r}\left\{[C^{\mathrm{eff}}(r) - C^{\mathrm{i}}(r)] + \frac{\alpha_C^{\mathrm{i}}(r)}{C^{\mathrm{i}}(r)}[C^{\mathrm{eff}}(r) - C^{\mathrm{i}}(r)]^2\right\} \tag{7}$$

where $m = 3$ (or 2) for the case of a composite sphere (or cylinder), and the initial condition of (7) can be given as $C^{\mathrm{eff}}(r_0) = C^{\mathrm{f}}$.

3.2. The strain energy changes of an infinite matrix

Due to the presence of a composite sphere/cylinder, the strain energy changes Δf_C of an infinite matrix that is subjected to the five far-field stress systems respectively can be obtained by replacing the composite sphere/cylinder with the effective homogeneous particle/fiber whose properties are given by the differential equation (7),

$$\Delta f_C = -\frac{1}{2}V^{\mathrm{f}}\frac{(\sigma^{\infty})}{C^{\mathrm{m}}}\frac{C^{\mathrm{eff}}(r_1) - C^{\mathrm{m}}}{C^{\mathrm{m}} + \alpha_C^{\mathrm{m}}[C^{\mathrm{eff}}(r_1) - C^{\mathrm{m}}]} \tag{8}$$

Note that (8) is exact for Δf_K, $\Delta f_{K_{23}}$ and $\Delta f_{G_{12}}$ since the replacement method is exact for any layered interphases for these cases. But it is not exact for Δf_G and $\Delta f_{G_{23}}$. Therefore, its validity needs to be verified for these two cases.

3.3. Comparing with FEM results

For simplicity, the strain energy changes for the cases corresponding to the plane-strain bulk modulus and transverse shear modulus of the composite cylinder are now analyzed using FEM and compared with the present model. The analysis for the case of plane-strain bulk modulus may be used as a double check between the FEM model and the exact solution. The relatively thick interphase (the radii of the fiber and the composite cylinder are $r_0 = 0.8$ and $r_1 = 1$) is chosen to illustrate problems better. The plane strain problems of an infinite matrix containing a composite cylinder subjected to the far-field hydrostatic stresses with $\sigma_{22}^{\infty} = \sigma_{33}^{\infty} = 1$ and $\tau_{23} = 0$ and the far-field tension–compression stresses with $\sigma_{22}^{\infty} = 1$, $\sigma_{33}^{\infty} = -1$ and $\tau_{23} = 0$ (pure shear) are analyzed using FEM.

The Poisson's ratios of all constituents are taken as 0.25 in the comparison examples, and the transverse Young's modulus variation of the inhomogeneous interphase $E_{\mathrm{T}}^{\mathrm{i}}(r)$ is described as

$$\frac{E_{\mathrm{T}}^{\mathrm{i}}(r)}{E_{\mathrm{T}}^{\mathrm{m}}} = 1 - D\left[\frac{r_1 - r}{r_1 - r_0}\right]^Q \tag{9}$$

where $E_{\mathrm{T}}^{\mathrm{m}}$ is the transverse Young's modulus of the matrix, D and Q are material parameters. It is noted that $D = [E_{\mathrm{T}}^{\mathrm{i}}(r_0) - E_{\mathrm{T}}^{\mathrm{m}}]/E_{\mathrm{T}}^{\mathrm{m}}$ is similar to the damage parameter in the interphase variation model by Lutz and Zimmerman (1996). Figs. 1–4 plot some typical examples of the interphase variation model, in which a harder fiber with $E_{\mathrm{T}}^{\mathrm{f}}/E_{\mathrm{T}}^{\mathrm{m}} = 10$ and a softer fiber with $E_{\mathrm{T}}^{\mathrm{f}}/E_{\mathrm{T}}^{\mathrm{m}} = 0.2$ are considered. These material combinations of matrix, fiber and interphase will be analyzed using FEM and compared with the present analytical model (7) and (8).

An infinite matrix containing one composite cylinder with the interphase variations shown in Figs. 1–4 is analyzed using FEM. As the strain energy changes $\Delta f_{K_{23}}$ and $\Delta f_{G_{23}}$ are just concerned, the problem can be modeled as a plane strain problem. As it is impossible to simulate an infinite matrix using FEM, a sufficient large square, say 20 times larger than the centered composite cylinder, is considered to approximate the infinite matrix. The inhomogeneous interphase is approximated as 20 homogeneous layers in the finite

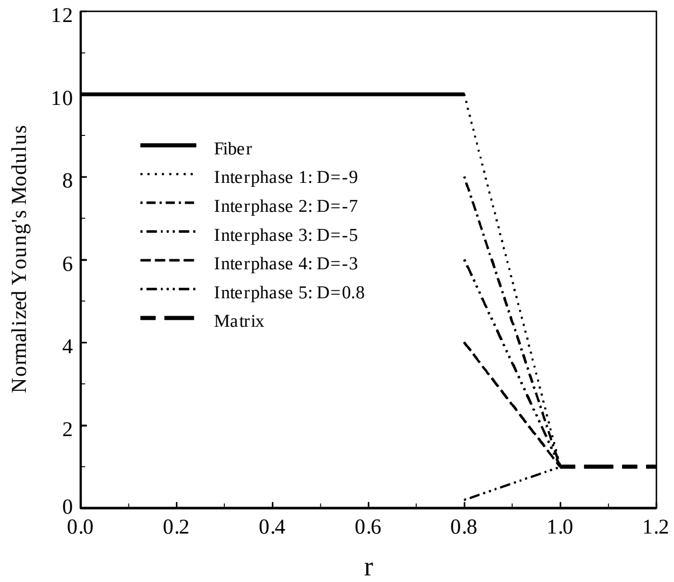

Fig. 1. Linear variations of normalized Young's modulus of interphase with material parameter $Q = 1$ and various D for a harder particle/fiber.

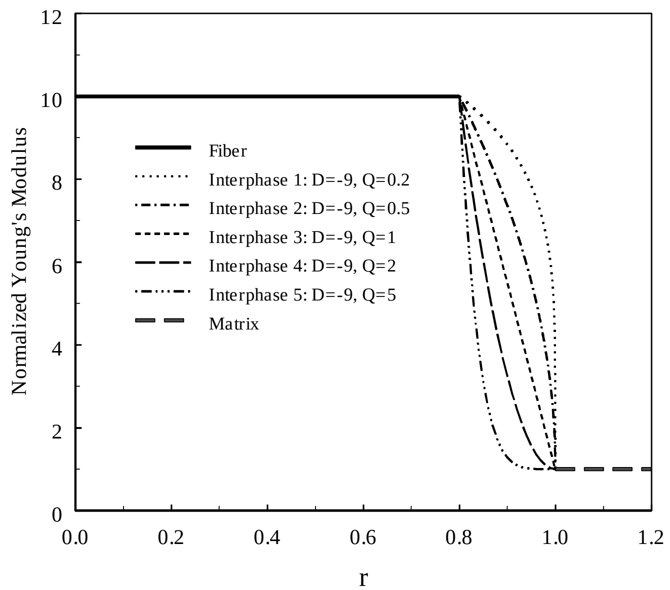

Fig. 2. Variations of normalized Young's modulus of interphase between matrix and fiber for a harder particle/fiber.

element model, and the transverse Young's modulus of each layer is taken as the average value of the corresponding inhomogeneous layer. It has been checked by comparing with more layers that 20 layers are sufficient to get convergent values when the relative error is set at 1%.

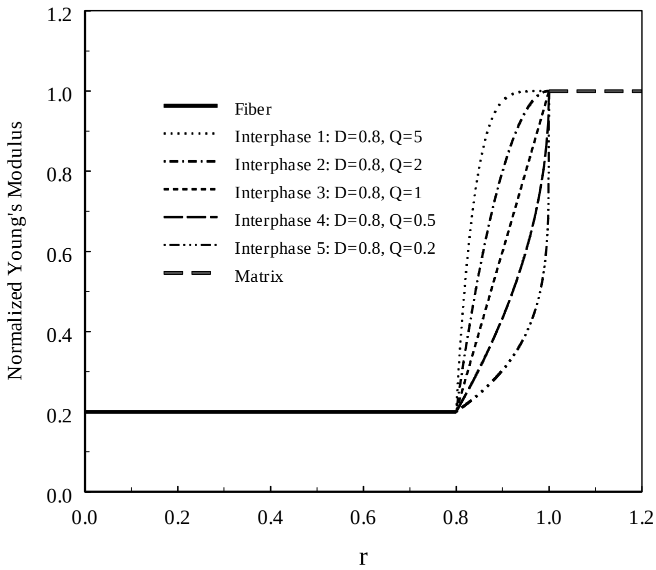

Fig. 3. Variations of normalized Young's modulus of interphase between matrix and particle/fiber for a softer particle/fiber.

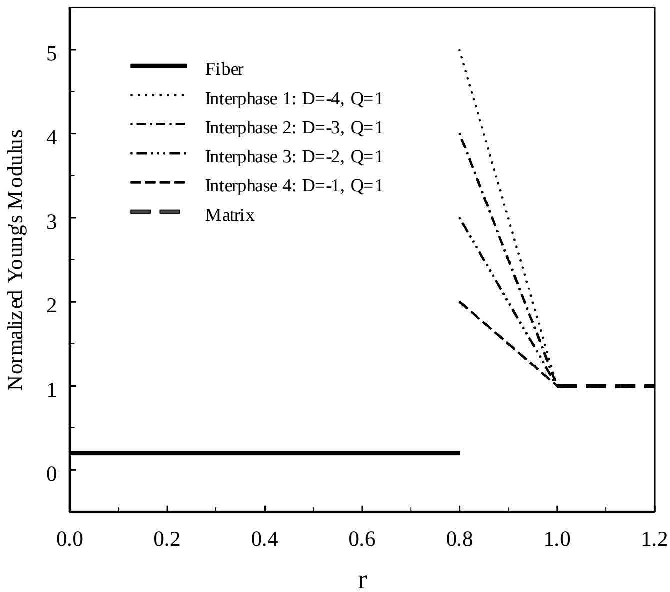

Fig. 4. Linear variations of normalized Young's modulus of interphase which are harder than the matrix and the particle/fiber.

Furthermore, the variations of the strain energy change $\Delta f_{K_{23}}$ and $\Delta f_{G_{23}}$ with the material parameter D and Q corresponding to the material combinations shown in Figs. 1–4 are solved using the present model, i.e., (7) and (8). These analytical results are plotted in Figs. 5–8 together with the FEM results, in which the $\Delta f_{K_{23}}$ and $\Delta f_{G_{23}}$ have been normalized by $\Delta f_{G,0}$, which is the strain energy change when the material properties of the interphase are taken as those of the matrix, that is

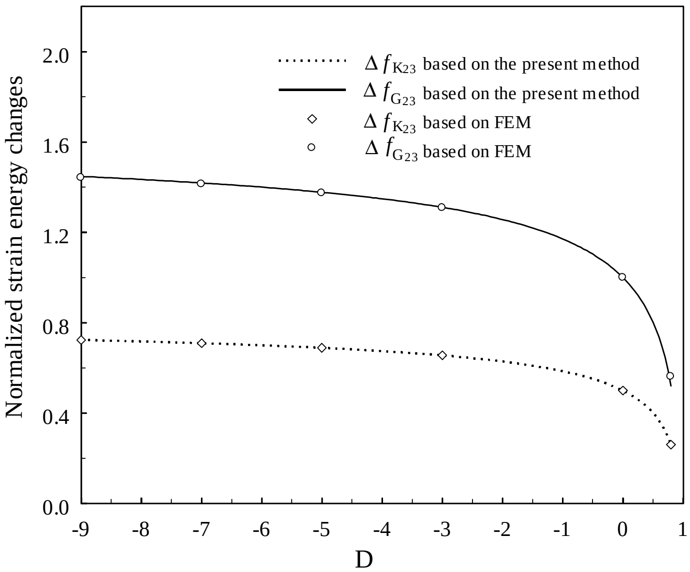

Fig. 5. Comparisons of strain energy changes between the predictions by the present model and the FEM results for the material combinations shown in Fig. 1.

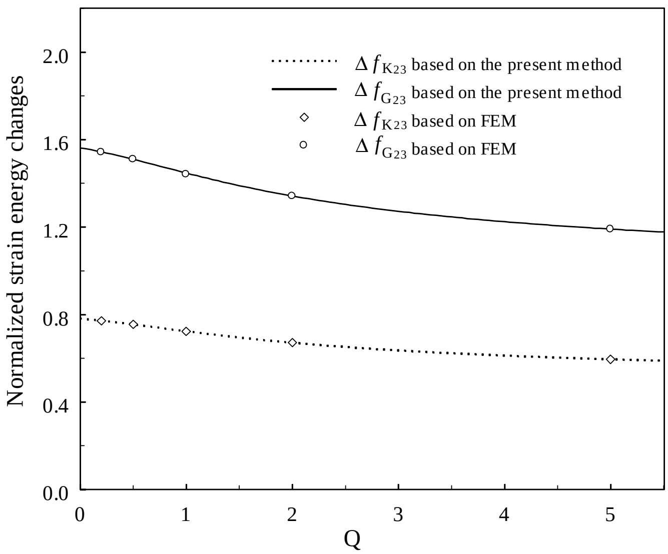

Fig. 6. Comparisons of strain energy changes between the predictions by the present model and the FEM results for the material combinations shown in Fig. 2.

$$\Delta f_{G_{23},0} = -\frac{1}{2}\pi r_0^2 \frac{\sigma^2}{G_{23}^{\mathrm{m}}} \frac{G_{23}^{\mathrm{f}} - G_{23}^{\mathrm{m}}}{G_{23}^{\mathrm{m}} + \alpha_{G_{23}}^{\mathrm{m}}(G_{23}^{\mathrm{f}} - G_{23}^{\mathrm{m}})} \tag{10}$$

It is seen from Figs. 5–8 that the FEM results are precisely consistent with the present model for the transverse bulk modulus. Thus, the FEM model, including its meshes and the interphase material model where the inhomogeneous interphase is approximated by 20 homogeneous layers, can be assumed valid.

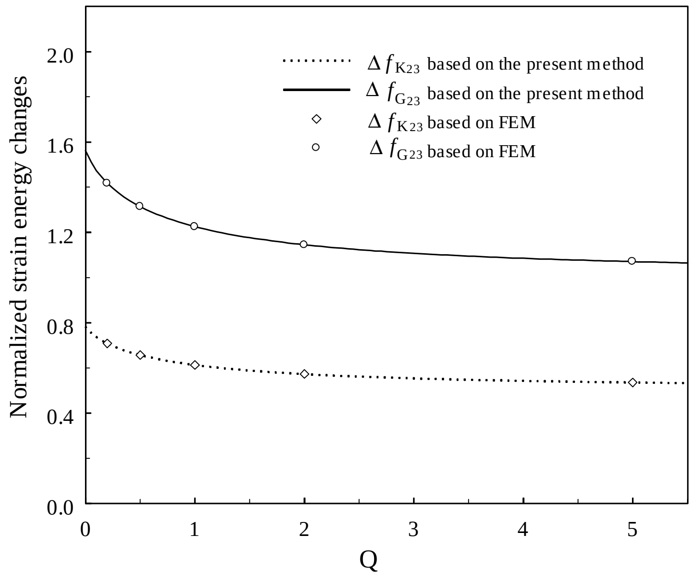

Fig. 7. Comparisons of strain energy changes between the predictions by the present model and the FEM results for the material combinations shown in Fig. 3.

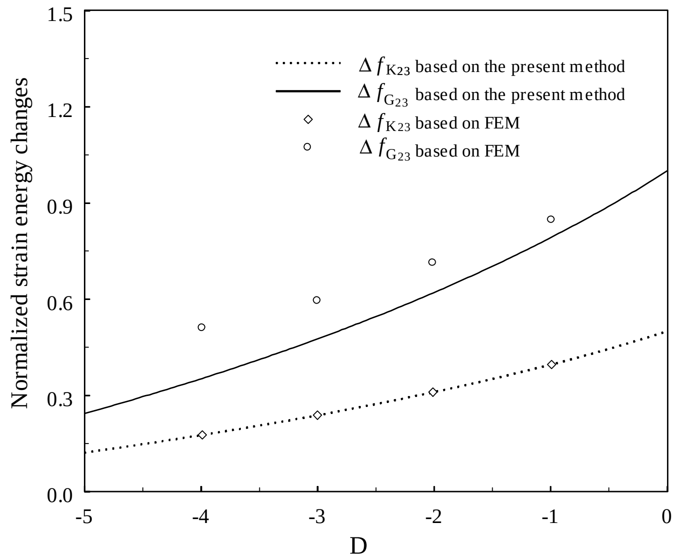

Fig. 8. Comparisons of strain energy changes between the predictions by the present model and the FEM results for the material combinations shown in Fig. 4.

For the case of the transverse shear modulus, the validity of the present model is dependent on the material properties of matrix, inclusion and interphase. For a rather damaged interphase with the damage parameter $D = 0.8$, the present model can still lead to satisfactory results, as shown in Fig. 5 in which the result of $\Delta f_{G_{23}}$ based on the present method is 0.521, while the FEM gives 0.561. Their relative difference is

7.7%. And if the Young's modulus of an interphase lies between the fiber and matrix, the present model is very accurate, as shown in Figs. 6 and 7. However, the model is not satisfactory for the material combinations with a softer fiber and harder interphase, as shown in Figs. 4 and 8. Therefore, the validity of the present model is limited to the kinds of material combinations shown in Figs. 1–3.

4. Effect of interactions among composite cylinders

The present model is exact or rather accurate for the single inclusion problems. Besides, it is exact for the moduli (except the two shear cases) of the ideal composite that is a collection of composite spheres/cylinders that progressively fill all the space. However, the model is not exact for practical composites when the interaction effects among composite spheres/cylinders take place, even for the case of bulk moduli. Therefore, its validity needs to be investigated further. For this purpose, the plane strain problem of an infinite matrix containing two composite cylinders and the composites with equilateral triangular distribution of composite cylinders are studied.

4.1. An infinite matrix containing two composite cylinders

The three typical material combinations are taken, i.e., $E^{\mathrm{f}}_{\mathrm{T}}/E^{\mathrm{m}} = 10$, $D = -9$, $Q = 1$; $E^{\mathrm{f}}_{\mathrm{T}}/E^{\mathrm{m}} = 0.2$, $D = 0.8$, $Q = 1$; and $E^{\mathrm{f}}_{\mathrm{T}}/E^{\mathrm{m}} = 10$, $D = 0.8$, $Q = 1$ as shown by the interphase 1 in Fig. 1; the interphase 3 in Fig. 3; and the interphase 5 in Fig. 1, respectively. The radii of fiber and composite cylinder are still 0.8 and 1. The two far-field stresses are hydrostatic stress with $\sigma_{22} = \sigma_{33} = 1$ and $\tau_{23} = 0$ and the tension–compression stresses with $\sigma_{22} = 1$, $\sigma_{33} = -1$ and $\tau_{23} = 0$. The centers of the two composite cylinders are located along the direction 3. The space between the two composite cylinders is set at 1% of their diameter, i.e., the distance of the two centers is 2.02. Due to the symmetry, a quarter of model can be used in FEM analysis, in which 200 nodes are set along the half peripheral of the composite cylinder and the inhomogeneous interphase is also approximated as 20 homogeneous layers. The relative errors have been controlled smaller than 1% comparing with finer meshes.

The corresponding problems by replacing the two composite cylinders with the two effective homogeneous fibers are also analyzed using FEM based on the same mesh for the comparison. The normalized strain energy changes $\Delta f_{K_{23}}$ and $\Delta f_{G_{23}}$ per composite cylinder or effective fiber are listed in Table 1.

It is seen from Table 1 that the present model is still rather accurate for the material combinations 1 and 2, i.e., the cases that the Young's modulus of interphase lies between those of fiber and matrix. But for the material combination 3, the relative differences due to the replacement are −12.0% and 15.4% for $\Delta f_{K_{23}}$ and $\Delta f_{G_{23}}$, while they are zero and 7.7% for the case of one single composite cylinder. Note that the small space

Table 1
Comparisons between the strain energy changes due to two composite cylinders (CC) and two corresponding effective homogeneous fibers (EF)

$\Delta f_K/\Delta f_{G,0}$			$\Delta f_G/\Delta f_{G,0}$		
CC	EF	(CC – EF)/EF (%)	CC	EF	(CC – EF)/EF (%)
Material combination 1: $E^f_T/E^m = 10$, $D = -9$, $Q = 1$					
0.762	0.755	0.9	1.637	1.606	1.9
Material combination 2: $E^f_T/E^m = 0.2$, $D = 0.8$, $Q = 1$					
0.643	0.639	0.6	1.197	1.212	−1.2
Material combination 3: $E^f_T/E^m = 10$, $D = 0.8$, $Q = 1$					
0.257	0.292	−12.0	0.614	0.532	15.4

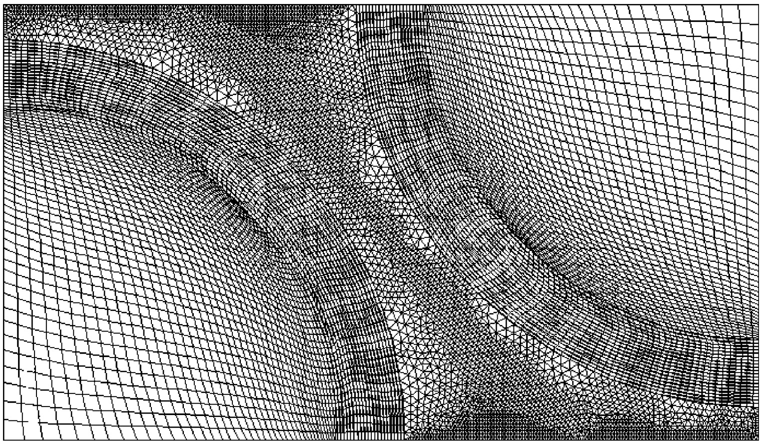

Fig. 9. FEM mesh of a unit cell where the radius of fiber, the thickness of interphase and the distance between the composite cylinders are 1, 0.2 and 0.2 (the volume fraction of the composite cylinders is 0.773 or that of fiber is 0.537).

between the two composite cylinders has been taken in the calculation example to illustrate the problem better. It is believed that the relative differences will be closer to those of one composite cylinder when the space between the composite cylinders becomes larger. Therefore, it can be assumed that the replacement is approximately acceptable for practical composites with the material combination 3 when the interaction effects among composite cylinders take place.

4.2. Equilateral triangular distribution of composite cylinders

The composite with equilateral triangular distribution of cylinders is taken to verify the validity of the present model further. Fig. 9 shows the FEM mesh of a unit cell, where the radius of fiber, the thickness of interphase and the distance between the composite cylinders are 1, 0.2 and 0.2, respectively. Thus, the volume fraction of the composite cylinders is 0.773 or that of fiber is 0.537 for the example. The previous three material combinations are considered again. Besides, the two composites designated as material combinations 4 and 5 that have been analyzed by Jasiuk and Kouider (1993), i.e., the glass/epoxy composite with elastic properties $E^m = 3.4$ GPa, $v^m = 0.38$; $E^f = 69.0$ GPa, $v^f = 0.20$ and the graphite/epoxy composite with elastic properties $E^m = 3.5$ GPa, $v^m = 0.35$; $E^f = 14.0$ GPa, $v^f = 0.20$ are also considered. In terms of the interphase model of Jasiuk and Kouider (1993), the interphase Young's modulus and Poisson's ratio are given by $E^i(r) = Pr^Q$ where P and Q are determined by assuming that the Young's modulus of interphase changes from that of matrix to that of fiber and $v^i = (v^m + v^f)/2$.

The effective plane strain bulk and transverse shear moduli for the five material combinations are obtained using FEM to analyze the unit cell. It is seen from the results listed in Table 2 that the present model is very satisfactory.

5. Effective moduli

5.1. Simple expressions for effective moduli

In terms of the generalized non-interacting solution to account for the presence of other particles or fibers, the effective moduli $\overline{K}$, $\overline{G}$ or $\overline{G}_{12}$, $\overline{K}_{23}$, $\overline{G}_{23}$ of particle or fiber-reinforced composites with an inhomogeneous interphase can be uniformly evaluated as

Table 2
Normalized effective moduli of the composites with equilateral triangular distribution of composite cylinders and effective fibers for the five material combinations (MC)

MC	Plane strain bulk modulus			Transverse shear modulus		
	CC	EF	(CC – EF)/EF (%)	CC	EF	(CC – EF)/EF (%)
1	3.660	3.671	−0.3	3.688	3.642	1.3
2	0.389	0.388	0.3	0.373	0.379	−1.6
3	1.661	1.662	−0.1	1.710	1.665	2.7
4	2.697	2.708	−0.4	4.679	4.530	3.3
5	1.609	1.580	1.8	2.414	2.400	0.6

$$\frac{\overline{C}}{C^{\mathrm{m}}} = 1 + \frac{\phi}{C^{\mathrm{m}}/[C^{\mathrm{eff}}(r_1) - C^{\mathrm{m}}] + (1-\phi)\alpha_C^{\mathrm{m}}} \tag{11a}$$

$$\frac{\mathrm{d}C^{\mathrm{eff}}(r)}{\mathrm{d}r} = -\frac{m}{r}\left\{[C^{\mathrm{eff}}(r) - C^{\mathrm{i}}(r)] + \frac{\alpha_C^{\mathrm{i}}(r)}{C^{\mathrm{i}}(r)}[C^{\mathrm{eff}}(r) - C^{\mathrm{i}}(r)]^2\right\} \quad \text{with} \quad C^{\mathrm{eff}}(r_0) = C^{\mathrm{f}} \tag{11b}$$

Furthermore, the effective longitudinal Young's modulus and major Poisson's ratio are also given based on the procedure of the present model and the rule of mixtures, that is,

$$\overline{E}_{11} = \phi E_{11}^{\mathrm{eff}} + (1-\phi)E_{11}^{\mathrm{m}} \tag{12a}$$

$$E_{11}^{\mathrm{eff}} = \frac{1}{r_1^2}\int_{r_0}^{r_{\mathrm{f}}} 2rE_{11}^{\mathrm{i}}(r)\,\mathrm{d}r + \frac{r_0^2}{r_1^2}E_{11}^{\mathrm{f}} \tag{12b}$$

$$\bar{v}_{12} = \phi v_{12}^{\mathrm{eff}} + (1-\phi)v_{12}^{\mathrm{m}} \tag{13a}$$

$$v_{12}^{\mathrm{eff}} = \frac{1}{r_1^2}\int_{r_0}^{r_{\mathrm{f}}} 2rv_{12}^{\mathrm{i}}(r)\,\mathrm{d}r + \frac{r_0^2}{r_1^2}v_{12}^{\mathrm{f}} \tag{13b}$$

where ϕ is the volume fraction of composite spheres/cylinders.

5.2. Comparing with existing models

5.2.1. Comparing with the model of Lutz and Zimmerman (1996)

Lutz and Zimmerman (1996) analyzed the bulk modulus of spherical particle-reinforced composites with the elastic moduli continuously varying with radius throughout the entire region outside the particle. By solving the radially symmetric problem of an infinite body containing a spherical particle and using the Mori–Tanaka method (Mori and Tanaka, 1973) to consider the presence of other particles, they gave a series solution for the effective bulk modulus. In their calculated example, the Poisson's ratios of all constituents are taken as 0.25, the normalized Young's modulus of particles is $E^{\mathrm{p}}/E^{\mathrm{m}} = 5$, and that of interphase region is expressed as

$$\frac{E^{\mathrm{i}}(r)}{E^{\mathrm{m}}} = 1 - D\left(\frac{r}{r_0}\right)^{-\beta} \tag{14}$$

For the case of $\beta = 10$ and various damage parameter D from −0.75 to 0.75, they plotted the effective bulk modulus variation with the volume fraction of particles.

For the case in which no clear interphase between interface zone and matrix exists, an alternative form of the present solution (11a) can be used, that is,

$$\frac{\overline{C}}{C^{\mathrm{m}}} = 1 - \frac{2\phi_0 C^{\mathrm{m}} \Delta f_C / V_0}{1 + 2\phi_0 \alpha_C^{\mathrm{m}} C^{\mathrm{m}} \Delta f_C / V_0} \tag{15}$$

with

$$\Delta f_C = -\frac{1}{2} V \frac{1}{C^{\mathrm{m}}} \frac{C^{\mathrm{eff}}(r_\infty) - C^{\mathrm{m}}}{C^{\mathrm{m}} + \alpha_C^{\mathrm{m}}(C^{\mathrm{eff}}(r_\infty) - C^{\mathrm{m}})} \tag{16}$$

where ϕ_0 is the volume fraction of the particles or fibers, V_0 and V are the volumes of one particle/fiber and one composite sphere/cylinder, and Δf_C is the strain energy change of the infinite matrix subjected to the far-field unit stresses corresponding to the modulus C due to the presence of one particle or fiber with the surrounding interphase zone; r_∞ should be sufficiently large to achieve the convergent Δf_C. Note that $\phi = \phi_0 V/V_0$ and (11a) can be recovered by substituting (16) into (15). The alternative form of (15) and (16) is convenient for those interphase regions that do not have a clear terminal in the matrix, such as the interphase variation model introduced by Lutz and Zimmerman (1996). The effective bulk modulus is solved using (11b), (15) and (16) for the example and plotted in Fig. 10. It is seen that the present results in Fig. 10 for the effective bulk modulus are precisely consistent with those in Fig. 4 of Lutz and Zimmerman (1996), as expected. Besides, the effective shear modulus is similarly obtained and plotted in Fig. 11. Note that it is very difficult to find out the effective shear modulus by solving the corresponding elastic problem as done by Lutz and Zimmerman (1996) for the case of effective bulk modulus.

5.2.2. Comparing with the results of Jasiuk and Kouider (1993)

Jasiuk and Kouider (1993) considered the two inetrphase variation models for unidirectional fiber reinforced composites, i.e., the power variation $E^{\mathrm{i}}(r) = Pr^Q$ with constant Poisson's ratio, and the linear variation $E^{\mathrm{i}}(r) = Pr + Q$, $v^{\mathrm{i}}(r) = Sr + T$, where P, Q, S and T are constants that are determined by the

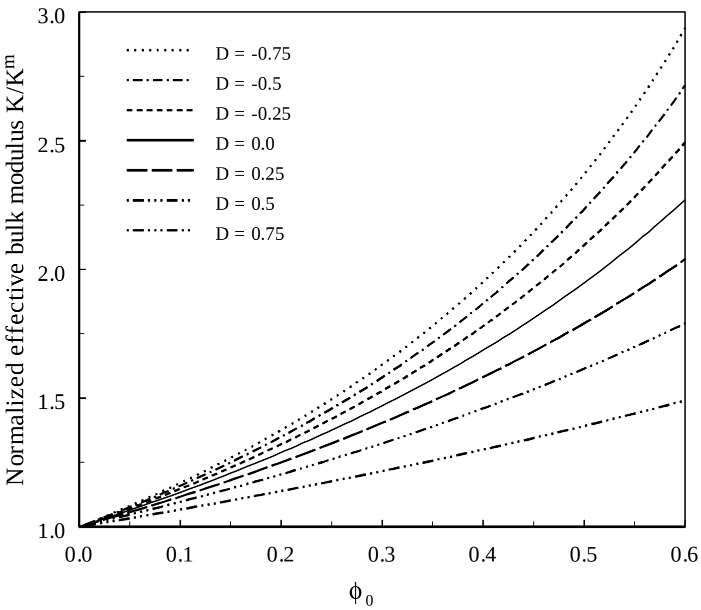

Fig. 10. Effective bulk modulus predicted by the present model coincides with those by Lutz and Zimmerman's model (1996).

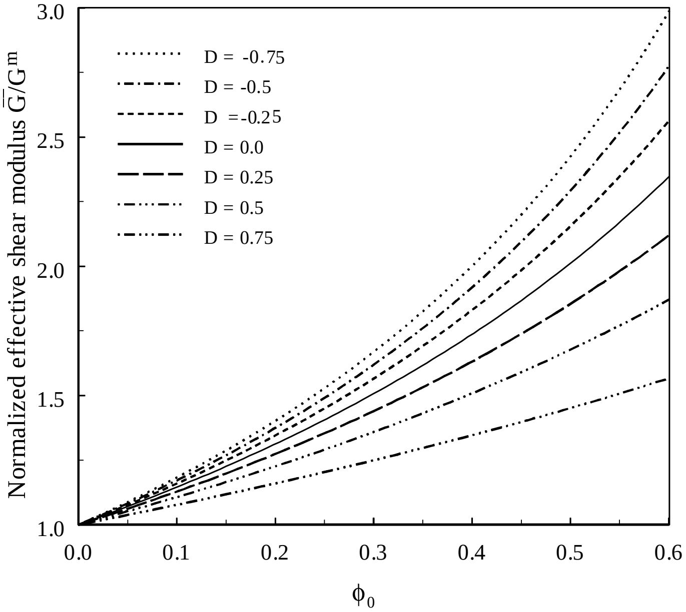

Fig. 11. Effective shear modulus predicted by the present model for the composite with the interphase variation by Lutz and Zimmerman (1996).

elastic properties at two ends of interphase zone. In their numerical examples, these constants are determined by assuming that the properties of interphase change from those of matrix to those of fiber.

Jasiuk and Kouider (1993) obtained the effective moduli by numerically solving the complicated governing differential equation for the elastic problem of inhomogeneous isotropic material with properties changing radially. They used the generalized self-consistent method (Christensen and Lo, 1979) to account for the presence of other composite cylinder for the effective transverse shear modulus, and the composite cylinder assemblage model (Hashin and Rosen, 1964) for other moduli.

The cases of plane strain bulk and transverse shear moduli are now taken to compare with the present model. The graphite/epoxy composite with elastic properties $E^m = 3.5$ GPa, $\nu^m = 0.35$, $E^f_T = 14.0$ GPa, $\nu^f_T = 0.20$ (Sottos et al., 1989) for the effective transverse shear modulus and the glass/epoxy composite with elastic properties $E^m = 3.4$ GPa, $\nu^m = 0.38$, $E^f = 69.0$ GPa, $\nu^f = 0.20$ for the effective plane strain bulk modulus were considered in Jasiuk and Kouider's calculated examples. Besides, the Young's modulus and Poisson's ratio of interphase were assumed as the power variation changing from that of the fiber to that of the matrix and the average of those of fiber and matrix, respectively. The normalized thickness of interphase, i.e., the ratio of interphase thickness and the radius of fiber, was taken as 0.1 and 0.2, respectively.

In terms of the present model, the composite cylinder is first assumed as an effective homogeneous fiber such that its elastic properties in transverse plane are determined using (11b). The transverse Young's modulus and Poisson's ratio can be derived from the plane strain bulk and transverse shear moduli. For the two interphase thicknesses, i.e., 0.1 and 0.2, they are solved as $E^{eff}_T = 11.974$ GPa, $\nu^{eff}_{23} = 0.220$ and $E^{eff}_T = 10.651$ GPa, $\nu^{eff}_T = 0.232$ corresponding to the graphite/epoxy composite, and $E^{eff}_T = 40.446$ GPa, $\nu^{eff}_T = 0.239$ and $E^{eff}_T = 29.640$ GPa, $\nu^{eff}_T = 0.253$ corresponding to the glass/epoxy composite. Then, the effective plane strain bulk modulus of the composites is obtained using (11a), i.e., the Mori–Tanaka method, which is equivalent to the composite cylinders assemblage-method of Hashin and Rosen (1964). To compare with Jasiuk and Kouider's results for the effective transverse shear modulus, instead of using (11a), the generalized self-consistent method (Christensen and Lo, 1979) is adopted.

It is expected that Jasiuk and Kouider's results and the present ones should be precisely consistent for the prediction of the effective bulk modulus and very agreeable to each other for the prediction of the transverse

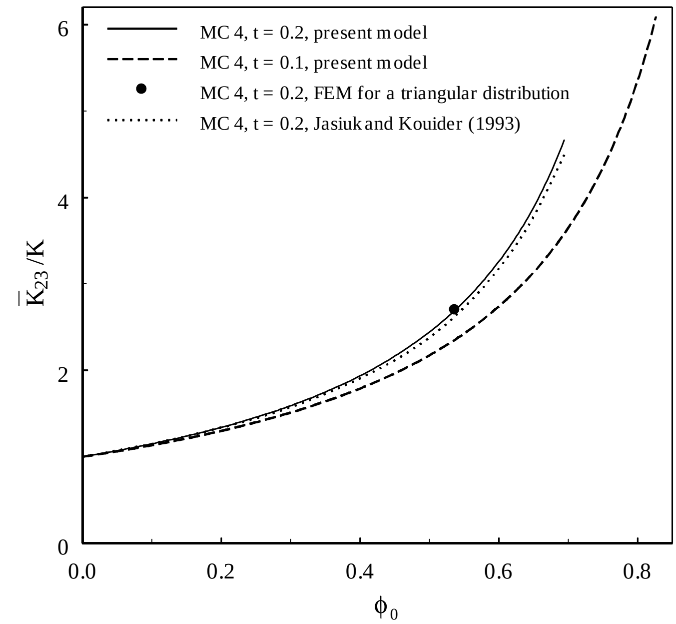

Fig. 12. Effective plane-strain bulk modulus predicted by the present model for the same case of material combination and geometries as done by Jasiuk and Kouider (1993).

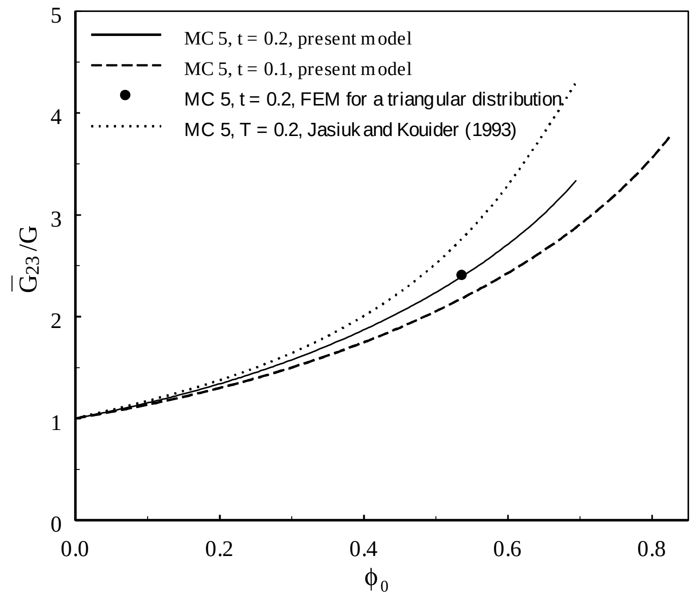

Fig. 13. Effective in-plane shear modulus predicted by the present model for the same case of material combination and geometries as done by Jasiuk and Kouider (1993).

shear modulus. However, the differences of the results predicted by the two models are significant, as shown in Figs. 12 and 13. For the effective plane strain bulk modulus, the present result is slightly higher than that of Jasiuk and Kouider, but for the effective transverse shear modulus, the present result is much lower than that of Jasiuk and Kouider. For a further verification, the effective moduli associated with equilateral triangular distribution of composite cylinders predicted by FEM, as listed in Table 2 are also plotted in Figs. 12 and 13. It is seen that FEM results support the present model. Besides, for the last points in the curves, the volume fraction of fibers is 0.6944 that corresponds to the volume fraction 1 of composite cylinders. Note that the normalized transverse shear modulus of fibers is 4.5, while the last point of Jasiuk and Kouider is about 4.3. This means that Jasiuk and Kouider's results are close to the results when the interphase properties are taken the same as those of the fibers. As Jasiuk and Kouider's approach is very complicated and troublesome, it is difficult to verify their calculation.

6. Summary

The replacement method (Hill, 1964) is extended to evaluate effective elastic moduli of composites reinforced by spherical particles or continuous cylindrical fibers, which involve an inhomogeneous interphase. The present model is highlighted by the independent governing differential equations for each modulus of the effective particle/fiber that is used to replace the composite sphere/cylinder, i.e., the particle/fiber together with its surrounding interphase. Then, the simple expressions for the effective elastic moduli are obtained by combining the governing differential equations with the Mori–Tanaka solution or the generalized non-interacting solution. The validity of the present model is verified by comparing with FEM results and some existing models, which are based on very complicated but rigorous derivations. It is revealed that the present model is rather accurate when the interphase properties vary between those of fiber and matrix. But it is not satisfactory when the interphase is much harder than the matrix and the particle/fiber. To be able to understand the reason of such a difference due to the range of the interphase properties with respect to that of the matrix and the particle/fiber, further study needs to be conducted.

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